PROJECTIVE CROSS-RATIO ON HYPERCOMPLEX NUMBERS

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Abstract. The paper presents a new cross-ratio of hypercomplex numbers based on projective geometry. We discuss the essential properties of the projective cross-ratio, notably its invariance under Möbius transformations. Applications to the geometry of conic sections and Möbius-invariant metrics on the upper half-plane are also given.

The cross-ratio is a number associated with four points, which is invariant under Möbius transformations. It has received some attention recently in the context of quarternions [3]. It is not well defined on hypercomplex numbers due to the presence of zero divisors, hence we introduce the projective cross-ratio to suit this situation.

1. Preliminaries

Firstly we look at definitions of hypercomplex numbers and their properties.

1.1. **Hypercomplex numbers.** Up to isomorphism, there are only three 2-dimensional commutative algebras with a unit over the real numbers. Each is isomorphic to one of the following hypercomplex numbers.

Definition 1.1. Define

- complex numbers $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i^2 = -1\}$;
- dual numbers $\mathbb{D} = \{a + b\epsilon : a, b \in \mathbb{R}, \epsilon^2 = 0\}$;
- double numbers $\mathbb{O} = \{a + bj : a, b \in \mathbb{R}, j^2 = 1\}$.

We use the notation $\mathbb{A} = \{a + b\iota : a, b \in \mathbb{R}\}$ to represent any of the above.

An important quantity is the respective modulus.

Definition 1.2. Let $z = x + y\iota \in \mathbb{A}$, we define the hypercomplex conjugate as $\bar{z} = x - y\iota$. Also the hypercomplex modulus is defined as $|z|^2 = z\bar{z}$.

Definition 1.3. $a \in \mathbb{A}$ is a zero divisor or $a \mid 0$ if and only if $a \neq 0$ and there exists $b \in \mathbb{A}$ such that $ab = 0$.

Remark 1.4. $z \in \mathbb{A}$ is a zero divisor, if and only if $|z|^2 = z\bar{z} = 0$. As the set of zero divisors for $\mathbb{C}, \mathbb{D}$ and $\mathbb{O}$ are $\emptyset, \{r\epsilon; r \in \mathbb{R}\}$ and $\{r(1 + j); r \in \mathbb{R}\} \cup \{r(1 - j); r \in \mathbb{R}\}$ respectively.

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1.2. **Projective geometry.** The projective cross-ratio is defined on a projective space.

Definition 1.5. $\lambda \in \mathbb{A}$ is unit if it has a multiplicative inverse.

Definition 1.6. Given $x, y, u, v \in \mathbb{A}$ and not both x, y or u, v are zero. Then $(x, y) \sim (u, v)$ if and only if there exists a unit $\lambda \in \mathbb{A}$ such that $(x, y) = (\lambda u, \lambda v)$.

The hypercomplex projective space is defined as follows.

Definition 1.7. The projective space is the set of equivalence classes of the equivalence relation $\sim$,

$$\mathbb{P}^1(\mathbb{A}) = \{[x, y] : x, y \in \mathbb{A} \text{ and } x \sim y\}. \tag{1.1}$$

The following is a definition of mapping from $\mathbb{A}$ to $\mathbb{P}^1(\mathbb{A})$ and the other way round.

Definition 1.8. The map $\mathfrak{S} : \mathbb{A} \to \mathbb{P}^1(\mathbb{A})$ is defined by $\mathfrak{S}(z) = [z, 1]$. Also the map $\mathfrak{P} : \{[x, y] : y \text{ does not divide } 0\} \mapsto \mathbb{A}$ is defined as $\mathfrak{P}((x, y)) = x/y$.

Two numbers z and w are "distinct" if $(z - w) \neq 0$.

Definition 1.9. Two points $(x_1, y_1), (x_2, y_2) \in \mathbb{P}^1(\mathbb{A})$ are essentially distinct if $x_1 y_2 - y_1 x_2$ is not a zero divisor or zero.

Define $\underline{\infty}, \underline{1}$ and $\underline{0} \in \mathbb{P}^1(\mathbb{A})$ as notation for the equivalence classes $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ respectively. In particular, for all 3 types of hypercomplex numbers $\underline{\infty} = [z, 0]$ such that z is not a zero divisor.

Remark 1.10. Given $\underline{z} \in \mathfrak{S}(\mathbb{A})$:
(1) $|\mathfrak{P}(\underline{z})| = |x|/|y|$,
(2) $\overline{\mathfrak{P}(\underline{z})} = \bar{x}/\bar{y}$.

Hence the following definition:

Definition 1.11. Given $\underline{z} = (x, y) \in \mathbb{P}^1(\mathbb{A})$
(1) Define conjugate on projective numbers by $\bar{\underline{z}} = [\bar{x}, \bar{y}]$
(2) Define modulus on projective numbers by $|\underline{z}|^2 = (|x|^2, |y|^2)$.

1.3. **Möbius Transformations.** Möbius transformations will play an important role in this paper.

Definition 1.12. A Möbius transformation is a function f, of hypercomplex variables $z \in \mathbb{A}$. It can be written in the form

$$f(z) = \frac{az + b}{cz + d}, \tag{1.2}$$

for some $a, b, c, d \in \mathbb{A}$, such that $ad - bc$ is a unit.

Definition 1.13. The General linear group is defined as $GL_2(\mathbb{A}) = \{A : det(A) \text{ a unit}\}$. The Projective linear group is the quotient group $PGL_2(\mathbb{A}) = GL_2(\mathbb{A})/\{\lambda I : \lambda \text{ a unit}\}$

Define a mapping by a matrix $A \in PGL_2(\mathbb{A})$ on $\underline{z} \in \mathbb{P}^1(\mathbb{A})$ by the usual vector multiplication.

Lemma 1.14. *$PGL_2(\mathbb{A})$ action on $\mathbb{P}^1(\mathbb{A})$ is well defined.*

Proof. True by linearity of $\sim$. $\square$

There is a group homomorphism from the group of Möbius transformations acting on $\mathbb{A}$ to the group $PGL_2(\mathbb{A})$ acting on $\mathfrak{S}(\mathbb{A})$.

Remark 1.15. Given a Möbius transformation $f(z)$, $x, y \in \mathbb{A}, y \nmid 0$.

$$\begin{aligned}\mathfrak{S}(f(x/y)) &= \mathfrak{S}\left(\frac{a(x/y)+b}{c(x/y)+d}\right)\\ &= \mathfrak{S}\left(\frac{ax+by}{cx+dy}\right)\\ &= \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}\\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.\end{aligned}$$

Hence the representation of Möbius transformation by $PGL_2(\mathbb{A})$.

Definition 1.16. From Yaglom's book [7, p277] we define a subset of $\mathbb{P}^1(\mathbb{A})$ as:

$$S = \{A\underline{z} : A \in GL_2(\mathbb{A}), \underline{z} \in \mathfrak{S}(\mathbb{A})\}/\{\lambda : \lambda \text{ a unit}\}. \tag{1.3}$$

Notice that S is invariant under $PGL_2(\mathbb{A})$.

Lemma 1.17. *S consists of the union of $\mathfrak{S}(\mathbb{A})$, $\{\underline{\infty}\}$ and one of the following:*

1. *for $\mathbb{C}$: the empty set,*
2. *for $\mathbb{D}$: the set $\left\{\begin{bmatrix} t \\ \epsilon \end{bmatrix}, t \in \mathbb{R}\right\}$,*
3. *for $\mathbb{O}$: the set*

$$\left\{\begin{bmatrix} t \\ 1+j \end{bmatrix}, t \in \mathbb{R}\right\} \cup \left\{\begin{bmatrix} t(1-j) \\ 1+j \end{bmatrix}, t \in \mathbb{R}\right\} \cup \left\{\begin{bmatrix} t \\ 1-j \end{bmatrix}, t \in \mathbb{R}\right\} \cup \left\{\begin{bmatrix} t(1+j) \\ 1-j \end{bmatrix}, t \in \mathbb{R}\right\}.$$

2. Projective Cross ratio

2.1. **Möbius transformations on $\mathbb{P}^1(\mathbb{A})$.**

The paper follows Alan Beardon's book [2, Chapter 13]. Differences are:

- In [2, Thm 13.2.1], "distinct" has been replaced by "essentially distinct".
- Möbius maps have been replaced by matrices from $PGL_2(\mathbb{A})$.
- Complex numbers have been replaced by members of $\mathbb{A}$ for use with projective points.

The following theorem relates to [2, Thm 13.2.1] and includes the definition of an important matrix.

Theorem 2.1. *Given two sets of pairwise essentially distinct points $\{\underline{z}_1, \underline{z}_2, \underline{z}_3\}$ and $\{\underline{w}_1, \underline{w}_2, \underline{w}_3\}$ from $\mathbb{P}^1(\mathbb{A})$, there exists a unique matrix $A \in PGL_2(\mathbb{A})$, such that $A\underline{z}_i = \underline{w}_i$ for all $i = 1, 2, 3$.*

Proof. Let $\underline{z}_i = [x_i, y_i], \underline{w}_j = [u_j, v_j] \in \mathbb{P}^1(\mathbb{A})$ and define $A \in PGL_2(\mathbb{A})$ as

$$A = \begin{pmatrix} (x_2y_3 - x_3y_2)y_1 & -(x_2y_3 - x_3y_2)x_1 \\ (x_2y_1 - x_1y_2)y_3 & -(x_2y_1 - x_1y_2)x_3 \end{pmatrix} \in PGL_2(\mathbb{A}). \tag{2.1}$$

The points are essentially distinct implies $\det(A) = (x_2y_1 - x_1y_2)(x_1y_3 - x_3y_1)(x_2y_3 - x_3y_2)$ does not divide zero, hence A^{-1} exists. A direct calculation shows $A\underline{z}_1 = \underline{0}, A\underline{z}_2 = \underline{1}, A\underline{z}_3 = \underline{\infty}$. Define $B = A'^{-1}$ where A' is the same as A with $x_i = u_i$ and $y_i = v_i$. Then define $M = BA$. M is the required matrix.

To prove the uniqueness, suppose that $M, N \in PGL_2(\mathbb{A})$ are distinct matrices, such that $M\underline{z}_i = \underline{w}_i = N\underline{z}_i$ for $i = 1, 2, 3$. Then $M^{-1}N$ fixes each $\underline{z}_i$. Let V be the matrix that maps $\underline{z}_1, \underline{z}_2, \underline{z}_3$ to $\underline{0}, \underline{1}, \underline{\infty}$ respectively. Then $A = V^{-1}M^{-1}NV$ is a matrix that fixes $\underline{0}, \underline{1}, \underline{\infty}$, this then must be the identity matrix. We can check this by solving the set of linear equations $A\underline{z} = \underline{z}$ for $\underline{z} = \underline{0}, \underline{1}, \underline{\infty}$. Hence as $PGL_2(\mathbb{A})$ is a group we have that $M = N$. □

Corollary 2.2. *If a matrix A fixes three pairwise essentially distinct points, then $A = I$.*

Proof. The theorem states that there exists a unique matrix which satisfies the above property. The identity matrix satisfies the property, so it is the unique matrix. □

2.2. **Cross-Ratio.**

Definition 2.3. The original cross-ratio from [2, p261] of four pairwise essentially distinct points z_1, z_2, z_3, z_4 in $\mathbb{A}$ is defined as

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}. \tag{2.2}$$

The Projective cross-ratio is constructed from the entries from the matrix (2.1). The properties from the original cross-ratio are replicated on the projective space.

Definition 2.4. The projective cross-ratio of four distinct points $\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4 \in \mathbb{P}^1(\mathbb{A})$ such that $\underline{z}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$, for $i \in \{1, 2, 3, 4\}$ is defined as:

$$[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4] = \begin{bmatrix} (x_1y_3 - x_3y_1)(x_2y_4 - x_4y_2) \\ (x_1y_2 - x_2y_1)(x_3y_4 - x_4y_3) \end{bmatrix} \in \mathbb{A}^2. \tag{2.3}$$

Four points are singular if their projective cross-ratio is not in S from Definition 1.16.

Note that for complex numbers the projective cross-ratio gives the same result:

Remark 2.5. For any pairwise distinct $z_1, z_2, z_3, z_4 \in \mathbb{C}$:

$$[z_1, z_2, z_3, z_4] = \mathfrak{P}([\mathfrak{S}(z_1), \mathfrak{S}(z_1), \mathfrak{S}(z_1), \mathfrak{S}(z_1)]) \tag{2.4}$$

where the left-hand side contains the original cross-ratio and the right-hand side, the projective one.

The following Lemmas link $\underline{0}, \underline{1}$ and $\underline{\infty} \in \mathbb{P}^1(\mathbb{A})$ to the projective cross-ratio.

Lemma 2.6. *Let $\underline{z} \in \mathbb{P}^1(\mathbb{A})$, then $[\underline{0}, \underline{1}, \underline{z}, \underline{\infty}] = \underline{z}$*

Proof. $[\underline{0}, \underline{1}, \underline{z}, \underline{\infty}] = \begin{bmatrix} (0 - x)(0 - 1) \\ (0 - 1)(0 - y) \end{bmatrix} = \underline{z}$ □

Lemma 2.7. *Given the matrix $A \in PGL_2(\mathbb{A})$ which maps $\underline{z}_1, \underline{z}_2, \underline{z}_4 \in \mathbb{P}^1(\mathbb{A})$ to $\underline{0}, \underline{1}, \underline{\infty}$ respectively, then $A\underline{z} = [\underline{z}_1, \underline{z}_2, \underline{z}, \underline{z}_4]$*

Proof. Calculate $A\underline{z}$. □

The next Theorem corresponds to [2, Thm 13.4.2]. It shows a necessary and sufficient condition for an existence of a matrix $A \in PGL_2(\mathbb{A})$ such that A maps four essentially distinct projective points to another four.

Theorem 2.8. *Given two sets of pairwise essentially distinct points $\underline{z}_i, \underline{w}_i \in \mathbb{P}^1(\mathbb{A})$, for $i = 1, 2, 3, 4$. We have the equality $[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4] = [\underline{w}_1, \underline{w}_2, \underline{w}_3, \underline{w}_4]$ if and only if there exists $A \in PGL_2(\mathbb{A})$ such that $A\underline{z}_i = \underline{w}_i$.*

Proof. For sufficiency say A is the required matrix and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\underline{z}_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$, $\underline{w}_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix}$, $A\underline{z}_i = \underline{w}_i$. Doing a substitution gives:

$$\begin{aligned} u_j v_i - u_i v_j &= (ax_j + by_j)(cx_i + dy_i) - (ax_i + by_i)(cx_j + dy_j) \\ &= (ad - bc)(x_j y_i - x_i y_j). \end{aligned}$$

Substitute this into the equation for the projective cross-ratio to give the required equivalence.

For necessity say $[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4] = [\underline{w}_1, \underline{w}_2, \underline{w}_3, \underline{w}_4]$. Let $H, G \in PGL_2(\mathbb{A})$ such that $G\underline{z}_1 = \underline{0}, G\underline{z}_2 = \underline{1}, G\underline{z}_4 = \underline{\infty}, H\underline{w}_1 = \underline{0}, H\underline{w}_2 = \underline{1}, H\underline{w}_4 = \underline{\infty}$. It then follows:

$$\begin{aligned} G\underline{z}_3 &= [\underline{0}, \underline{1}, G\underline{z}_3, \underline{\infty}] \\ &= [G\underline{z}_1, G\underline{z}_2, G\underline{z}_3, G\underline{z}_4] \\ &= [\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4] \\ &= [\underline{w}_1, \underline{w}_2, \underline{w}_3, \underline{w}_4] \\ &= [H\underline{w}_1, H\underline{w}_2, H\underline{w}_3, H\underline{w}_4] \\ &= [\underline{0}, \underline{1}, H\underline{w}_3, \underline{\infty}] \\ &= H\underline{w}_3 \end{aligned}$$

Define $F = H^{-1}G$, then $F\underline{z}_i = \underline{w}_i$ for each i. □

Proposition 2.9. *If $\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4 \in \mathbb{P}^1(\mathbb{A})$ are pairwise essentially distinct points then:*

(1) *The four points are non-singular.*
(2) $[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4]$ *is not* $\underline{0}, \underline{1}, \underline{\infty}$.
(3) $[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4]$ *is not in one of the sets from Lemma 1.17.*

Proof. For 1 and 3: As the $\underline{z}_i$ are pairwise essentially distinct, then none of the values of $x_i y_j - x_j y_i$ are 0 or divide zero. So if $[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4] = [u, v]$ then neither of u, v divide zero or are zero.

For 2: Let A be the matrix such that $A\underline{z}_1 = \underline{0}, A\underline{z}_2 = \underline{1}, A\underline{z}_4 = \underline{\infty}$ then $[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4] = A\underline{z}_3$. So if $[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4] = \underline{0}, \underline{1}$ or $\underline{\infty}$ then $A\underline{z}_3 = \underline{0}, \underline{1}$ or $\underline{\infty}$. However A is injective, hence a contradiction. □

3. Permutations

This section shows the dependence of the projective cross-ratio from cyclic permutations. The results correspond to [2, Sec 13.4].

Remark 3.1. The matrices

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}-1&1\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\-1&1\end{pmatrix}, \begin{pmatrix}1&-1\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\1&-1\end{pmatrix} \in PGL_2(\mathbb{A}) \tag{3.1}$$

permutate $\underline{0}, \underline{1}, \underline{\infty}$ and correspond to the permutations

$$(\underline{0})(\underline{1})(\underline{\infty}), (\underline{1})(\underline{0}\ \underline{\infty}), (\underline{0}\ \underline{1})(\underline{\infty}), (\underline{0}\ \underline{1}\ \underline{\infty}), (\underline{1}\ \underline{0}\ \underline{\infty}), (\underline{0})(\underline{1}\ \underline{\infty}), \tag{3.2}$$

respectively.

The members of (3.1) are a group closed under matrix multiplication. Each permutation of three points corresponds to a matrix in $PGL_2(\mathbb{A})$.

Proposition 3.2. *Given four non-singular pairwise distinct points* $\underline{z}_i \in \mathbb{P}^1(\mathbb{A})$, $i = 1, 2, 3, 4$ *and a permutation* $\rho \in S_4$. *If* $[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4] = \underline{\lambda}$ *for* $\underline{\lambda} \in \mathbb{P}^1(\mathbb{A})$, *then*

$$[\underline{z}_{\rho^{-1}(1)}, \underline{z}_{\rho^{-1}(2)}, \underline{z}_{\rho^{-1}(3)}, \underline{z}_{\rho^{-1}(4)}] = F_\rho \underline{\lambda}, \ \textit{for } F_\rho \in PGL_2(\mathbb{A}). \tag{3.3}$$

Proof. Let $\lambda = [\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4]$ and let $A \in GL_2(\mathbb{A})$ be the matrix such that $A\underline{z}_1 = \underline{0}$, $A\underline{z}_2 = \underline{1}$, $A\underline{z}_4 = \underline{\infty}$. By the invariance of the projective cross-ratio under $PGL_2(\mathbb{A})$, we see that $A\underline{z}_3 = \underline{\lambda}$. Now

$$[\underline{z}_{\rho^{-1}(1)}, \underline{z}_{\rho^{-1}(2)}, \underline{z}_{\rho^{-1}(3)}, \underline{z}_{\rho^{-1}(4)}] = [A\underline{z}_{\rho^{-1}(1)}, A\underline{z}_{\rho^{-1}(2)}, A\underline{z}_{\rho^{-1}(3)}, A\underline{z}_{\rho^{-1}(4)}], \tag{3.4}$$

is then the projective cross-ratio of $\underline{0}, \underline{1}, \underline{\lambda}$ and $\underline{\infty}$ in some order. Hence it only relies on ρ and $\underline{\lambda}$, so is of the form $F_\rho \underline{\lambda}$. □

Proposition 3.3. *The* $F_\rho \in PGL_2(\mathbb{A})$, *for* ρ *a transposition, is in* (3.1).

Proof. The proof is just calculation and so for an illustration we shall do one example, say $\rho = (12)$, $\lambda = \begin{bmatrix}u\\v\end{bmatrix}$ and a matrix A from (2.1) then

$$\begin{aligned}[\underline{z}_{\rho^{-1}(1)}, \underline{z}_{\rho^{-1}(2)}, \underline{z}_{\rho^{-1}(3)}, \underline{z}_{\rho^{-1}(4)}] &= [\underline{z}_2, \underline{z}_1, \underline{z}_3, \underline{z}_4] \\ &= [A\underline{z}_2, A\underline{z}_1, A\underline{z}_3, A\underline{z}_4] \\ &= [\underline{1}, \underline{0}, \underline{\lambda}, \underline{\infty}] \\ &= \begin{bmatrix}(v-u)(0-1)\\(1-0)(0-v)\end{bmatrix} \\ &= \begin{pmatrix}-1&1\\0&1\end{pmatrix}\lambda.\end{aligned}$$

Then for the others we have

$$\begin{aligned}F_{(12)} = F_{(34)} &= \begin{pmatrix}-1&1\\0&1\end{pmatrix}, \\ F_{(23)} = F_{(14)} &= \begin{pmatrix}0&1\\1&0\end{pmatrix}, \\ F_{(13)} = F_{(24)} &= \begin{pmatrix}1&0\\1&-1\end{pmatrix}.\end{aligned} \tag{3.5}$$

□

Proposition 3.4. *The matrices F_ρ from* (3.1) *have the property $F_{\sigma\rho} = F_\sigma F_\rho$, for $\sigma, \rho \in S_4$.*

Proof. Let $\mu = \sigma\rho$, $\underline{w}_{\rho(k)} = \underline{z}_k$ and $\underline{u}_{\sigma(j)} = \underline{w}_j$ then $\underline{u}_{\mu(k)} = \underline{u}_{\sigma(\rho(k))} = \underline{w}_{\rho(k)} = \underline{z}_k$ hence:

$$\begin{aligned} F_\sigma F_\rho[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4] &= F_\sigma[\underline{w}_1, \underline{w}_2, \underline{w}_3, \underline{w}_4] \\ &= [\underline{u}_1, \underline{u}_2, \underline{u}_3, \underline{u}_4] \\ &= [\underline{z}_{\mu^{-1}(1)}, \underline{z}_{\mu^{-1}(2)}, \underline{z}_{\mu^{-1}(3)}, \underline{z}_{\mu^{-1}(4)}] \\ &= F_\mu[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4] \end{aligned}$$

□

The previous propositions 3.2, 3.3, 3.4 are summarized in the following theorem. The theorem corresponds to [2, Thm 13.5.1].

Theorem 3.5. *For each $\rho \in S_4$ there is a matrix $F_\rho \in$*

$$\Gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \right\},$$

which permute $\{\underline{0}, \underline{1}, \underline{\infty}\}$, such that for any non-singular pairwise distinct $\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4$

$$[\underline{z}_{\rho^{-1}(1)}, \underline{z}_{\rho^{-1}(2)}, \underline{z}_{\rho^{-1}(3)}, \underline{z}_{\rho^{-1}(4)}] = F_\rho[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4]. \tag{3.6}$$

So $\rho \mapsto F_\rho$ is a homomorphism of S_4 onto Γ with kernel

$$K = \{I, (12)(34), (13)(24), (14)(23)\}. \tag{3.7}$$

4. Cycles

Cycles are natural objects invariant under Möbius transformations. They are defined by any of the following equivalent equations, [6], [5] [4]:

$$\begin{gathered} Kx\bar{x} - Lx\bar{y} - \bar{L}\bar{x}y + My\bar{y} = 0, \\ \begin{pmatrix} \bar{x} & \bar{y} \end{pmatrix} \begin{pmatrix} K & \bar{L} \\ -L & M \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0, \\ \begin{pmatrix} -\bar{y} & \bar{x} \end{pmatrix} \begin{pmatrix} L & -M \\ K & \bar{L} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0. \end{gathered}$$

Where $K = k\iota, M = m\iota, k, m \in \mathbb{R}, L \in \mathbb{A}$. Here is the projective version:

Definition 4.1. A cycle C is the set of points $\underline{z} \in \mathbb{P}^1(\mathbb{A})$ satisfying

$$\underline{\bar{z}}^T \begin{pmatrix} K & \bar{L} \\ -L & M \end{pmatrix} \underline{z} = 0, \tag{4.1}$$

for $K = k\iota, M = m\iota, k, m \in \mathbb{R}, L \in \mathbb{A}$ with cycle matrix $\mathbf{C} = \begin{pmatrix} K & \bar{L} \\ -L & M \end{pmatrix}$.

In Yaglom's book [7, p261], a cycle is defined to be the set of points satisfying $[z_1, z_2, z, z_4] = [\bar{z}_1, \bar{z}_2, \bar{z}, \bar{z}_4]$. This is the same as an equation $Az\bar{z} + Bz - \bar{B}\bar{z} + C = 0$, hence the following proposition.

Proposition 4.2. *Let $\underline{z}_1, \underline{z}_2, \underline{z}_4 \in \mathbb{P}^1(\mathbb{A})$ be fixed pairwise essentially distinct points. Then the set of $\underline{z} \notin \{\underline{z}_1, \underline{z}_2, \underline{z}_4\}$ satisfying $[\underline{z}_1, \underline{z}_2, \underline{z}, \underline{z}_4] = [\underline{\bar{z}}_1, \underline{\bar{z}}_2, \underline{\bar{z}}, \underline{\bar{z}}_4]$ with the points $\{\underline{z}_1, \underline{z}_2, \underline{z}_4\}$, is a cycle.*

Proof. $[\underline{z}_1, \underline{z}_2, \underline{z}, \underline{z}_4] = [\bar{\underline{z}}_1, \bar{\underline{z}}_2, \bar{\underline{z}}, \bar{\underline{z}}_4]$ is the same as $A\underline{z} = \bar{A}\bar{\underline{z}}$ from Lemma 2.7. Implying $\bar{A}^{-1}A\underline{z} = \bar{\underline{z}}$. As

$$\begin{bmatrix}-\bar{y} & \bar{x}\end{bmatrix}\begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix} = (0),$$

$$\text{then}\begin{bmatrix}-\bar{y} & \bar{x}\end{bmatrix}\bar{A}^{-1}A\begin{bmatrix}x\\ y\end{bmatrix} = (0),$$

$$\underline{z}^T\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\bar{A}^{-1}A\underline{z} = 0.$$

Let:

$$\begin{aligned}
L &= (x_2y_1 - x_1y_2)(\bar{x}_2\bar{y}_3 - \bar{x}_3\bar{y}_2)\bar{x}_1y_3\\
&\qquad -(\bar{x}_2\bar{y}_1 - \bar{x}_1\bar{y}_2)(x_2y_3 - x_3y_2)x_1\bar{y}_3,\\
K' &= (\bar{x}_2\bar{y}_1 - \bar{x}_1\bar{y}_2)(x_2y_3 - x_3y_2)x_1\bar{x}_3,\\
M' &= (\bar{x}_2\bar{y}_1 - \bar{x}_1\bar{y}_2)(x_2y_3 - x_3y_2)y_1\bar{y}_3. \text{ Hence:}\\
\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\bar{A}^{-1}A &= \frac{1}{det(\bar{A})}\begin{pmatrix}(K' - \bar{K}') & \bar{L}\\ -L & M' - \bar{M}'\end{pmatrix},
\end{aligned}$$

which is a cycle matrix. □

Corollary 4.3. *Any three pairwise essentially distinct points $\underline{z}_1, \underline{z}_2$ and $\underline{z}_4$ define a cycle, via Proposition 4.2. In particular, the $\underline{z}_i$ are in that cycle.*

Example 4.4. If cycle contains $\underline{0}, \underline{a}$ and $\underline{\infty}$, $a \in \mathbb{R}\backslash\{0\}$, then it is the real line. As

$$\begin{aligned}
\begin{bmatrix}0 & 1\end{bmatrix}\begin{pmatrix}K & \bar{L}\\ -L & M\end{pmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} &= M = 0,\\
\begin{bmatrix}1 & 0\end{bmatrix}\begin{pmatrix}K & \bar{L}\\ -L & M\end{pmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} &= K = 0,\\
\begin{bmatrix}a & 1\end{bmatrix}\begin{pmatrix}0 & \bar{L}\\ -L & 0\end{pmatrix}\begin{bmatrix}a\\ 1\end{bmatrix} &= a(\bar{L} - L) = 0,
\end{aligned}$$

$L \in \mathbb{R}$. Cycle matrix $\begin{pmatrix}0 & b\\ -b & 0\end{pmatrix}$, for $b \in \mathbb{R}\backslash\{0\}$, is then the matrix representing the real line. Similarly if $\underline{0}, \underline{a\iota}$ and $\underline{\infty}$, for $a \in \mathbb{R}\backslash\{0\}$, are in a cycle then it is the imaginary axis, with cycle matrix $\begin{pmatrix}0 & -b\iota\\ -b\iota & 0\end{pmatrix}$, $b \in \mathbb{R}\backslash\{0\}$.

Remark 4.5. From the known property of determinant

$$\begin{aligned}
det(\bar{A}^{-1}A) &= det(\bar{A}^{-1})det(A)\\
&= \frac{(x_2y_1 - x_1y_2)(x_2y_4 - x_4y_2)(x_1y_4 - x_4y_1)}{(\bar{x}_2\bar{y}_1 - \bar{x}_1\bar{y}_2)(\bar{x}_2\bar{y}_4 - \bar{x}_4\bar{y}_2)(\bar{x}_1\bar{y}_4 - \bar{x}_4\bar{y}_1)} = |L|^2 + KM.
\end{aligned}$$

Definition 4.6. A set of distinct points $\{\underline{z}_1, \underline{z}_2, ..., \underline{z}_n\} \subset \mathbb{P}^1(\mathbb{A})$ are concyclic if and only if they satisfy $\bar{\underline{z}}_k\mathbf{C}\underline{z}_k = 0$ for the same cycle matrix $\mathbf{C}$.

This definition provides the following theorem related to the results [7, p275. (42a)], [2, Thm 13.4.4].

Corollary 4.7. *Four pairwise distinct points $z_1, z_2, z_3, z_4 \in \mathbb{A}$, are concyclic in $\mathbb{A}$ if and only if $\mathfrak{S}(z_i)$ for all $i = 1, 2, 3, 4$ are also concyclic.*

Proof. Four distinct points in $\mathbb{A}$ are concylic if and only if the original cross-ratio of them is real, from [7, p275,(42a)]. Let $\underline{z}_i = \mathfrak{S}(z_i)$ for each i, and $\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4$ be concyclic in $\mathbb{P}^1(\mathbb{A})$. Then let $\underline{\lambda} = [\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4]$. Since $\underline{\lambda} = \bar{\underline{\lambda}}$ then $\mathfrak{P}(\underline{\lambda}) \in \mathbb{R}$. Hence due to 2.5 the original cross-ratio is real. So the points z_i are concyclic. The necessity follows similarly. If $z_i \in \mathbb{A}$ are concyclic, $\mathfrak{S}(z_i)$ are also concyclic. □

Definition 4.8. A cycle C, with cycle matrix $\mathbf{C}$, is mapped by a matrix $A \in PGL_2(\mathbb{A})$, to the cycle C' with cycle matrix $\mathbf{C}' = \bar{A}^T \mathbf{C} A$.

This leads to a proposition about points on two interconnecting cycles [2, Exercise 13.4], which only applies to $\mathbb{O}$ and $\mathbb{C}$.

Proposition 4.9. *For $\mathbb{A} = \mathbb{O}$ or $\mathbb{C}$: Given two distinct intersecting cycles C, C' and four pairwise essentially distinct points $\underline{z}_1, \underline{z}_2, , \underline{z}_3, \underline{z}_4 \in \mathbb{P}^1(\mathbb{A})$, such that $\underline{z}_1, \underline{z}_2, \underline{z}_4 \in C$ and $\underline{z}_1, \underline{z}_3, \underline{z}_4 \in C'$. The C and C' can be mapped to the real and imaginary axes by a matrix $A \in PGL_2(\mathbb{A})$ if and only if $[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4]$ is on the imaginary axis.*

Proof. Say $[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4]$ is on the imaginary axis, then let A be the matrix from (2.1) with $\underline{z}_3$ replaced by $\underline{z}_4$. Say A has mapped C and C' to R and I respectively. Notice $A\underline{z}_1 = \underline{0}, A\underline{z}_4 = \underline{\infty} \in R \cup I$, $A\underline{z}_2 = \underline{1} \in R$ and $[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4] = A\underline{z}_3 = [t\iota, 1] \in I$, for $t \in \mathbb{R}$. Then due to Example (4.4) we conclude R is the real axis and I is the imaginary axis.

Say there exists a matrix $A \in PGL_2(\mathbb{A})$, such that it maps C to the real axis and C' to the imaginary axis. As $\underline{z}_1$ and $\underline{z}_4$ are on C and C', $A\underline{z}_1$ and $A\underline{z}_4$ are both on the real and imaginary axis. So $A\underline{z}_1, A\underline{z}_4 \in \{\underline{0}, \underline{\infty}\}$. Say $A\underline{z}_2 = [r, 1]$ and $A\underline{z}_3 = [t\iota, 1], r, t \in \mathbb{R}$, then:

$$\begin{aligned} [\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4] &= [A\underline{z}_1, A\underline{z}_2, A\underline{z}_3, A\underline{z}_4], \\ &= \begin{cases} \text{if } A\underline{z}_1 = \underline{0} & [\underline{0}, [r,1], [t\iota, 1], \underline{\infty}] = [t\iota, 1], \\ \text{if } A\underline{z}_4 = \underline{0} & [\underline{\infty}, [r,1], [t\iota, 1], \underline{0}] = [1, t\iota] = [\frac{1}{t}\iota, \iota^2]. \end{cases} \end{aligned}$$

For both values of $A\underline{z}_1$, $[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4]$ is then on the imaginary axis. □

This suggests the following definition:

Definition 4.10. For $\mathbb{A} = \mathbb{O}$ or $\mathbb{C}$: Two cycles C, C' are projective-orthogonal if and only if there exists four pairwise distinct points $\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4$ such that $\underline{z}_1, \underline{z}_4 \in C \cap C'$, $\underline{z}_2 \in C, \underline{z}_3 \in C'$ and $[\underline{z}_1, \underline{z}_2, \underline{z}_3, \underline{z}_4]$ is on the imaginary axis.

This is then similar to cycle-orthogonality defined in [6, Sec 5.3] [5], given as:

Definition 4.11. The cycle product of two cycles C, C' with cycle matrices $\mathbf{C}, \mathbf{C}'$ is

$$\langle \mathbf{C}, \mathbf{C}' \rangle = -tr\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{C} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{\mathbf{C}}' \right), \tag{4.2}$$

where the r.h.s. is the product of matrices. The two cycles are then cycle-orthogonal if $\langle \mathbf{C}, \mathbf{C}' \rangle = 0$.

Lemma 4.12. *For all $A \in PGL_2(\mathbb{A})$ and cycles C and C',*

$$\langle \mathbf{C}, \mathbf{C}' \rangle = \det(A)^2 \langle \bar{A}^T \mathbf{C} A, \bar{A}^T \mathbf{C}' A \rangle. \tag{4.3}$$

Proof.

$$\begin{aligned}\langle \bar{A}^T\mathbf{C}A, \bar{A}^T\mathbf{C}'A\rangle &= -tr\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\bar{A}^T\mathbf{C}A\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}A^T\bar{\mathbf{C}}'\bar{A}\right),\\
&= -tr\left(\det(A)^2\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\bar{A}^{-1}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\mathbf{C}A\right.\\
&\left.\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}A^{-1}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\bar{\mathbf{C}}'\bar{A}\right),\\
&= -tr\left(\det(A)^2\bar{A}^{-1}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\mathbf{C}AA^{-1}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\bar{\mathbf{C}}'\bar{A}\right)\\
&= \det(A)^2\langle\mathbf{C},\mathbf{C}'\rangle.\end{aligned}$$

□

We can then show the equivalence of cycle-orthogonal and projective-orthogonal.

Theorem 4.13. *For $\mathbb{A} = \mathbb{O}$ or $\mathbb{C}$: Two distinct cycles C, C' are projective-orthogonal if and only if they are cycle-orthogonal.*

Proof. Say you can map C to the real line defined by the cycle matrix $\mathbf{R} = \begin{pmatrix}0 & l\\ -l & 0\end{pmatrix}$ for some $l \in \mathbb{R}$, and C' to the imaginary line defined by the cycle matrix $\mathbf{I} = \begin{pmatrix}0 & r\iota\\ r\iota & 0\end{pmatrix}, r \in \mathbb{R}$ both by the same matrix B. Then by Lemma 4.12

$$\begin{aligned}\langle\mathbf{C},\mathbf{C}'\rangle &= -\det(B)^2 tr(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\mathbf{R}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\bar{\mathbf{I}})\\
&= \det(B)^2(-lr\iota + lr\iota) = 0.\end{aligned}$$

Say $\langle\mathbf{C},\mathbf{C}'\rangle = 0$, and $\underline{z}_1, \underline{z}_2, \underline{z}_3 \in C$ are distinct points. Let $A \in PGL_2(\mathbb{A})$ be the matrix such that $A\underline{z}_1 = \underline{0}, A\underline{z}_2 = \underline{1}$ and $A\underline{z}_3 = \underline{\infty}$, then $\bar{A}^T\mathbf{C}A = \mathbf{R}$. Let $\bar{A}^T\mathbf{C}'A = \begin{pmatrix}K' & \bar{L}'\\ -L' & M'\end{pmatrix}$, then by Lemma 4.12:

$$\begin{aligned}\langle\mathbf{C},\mathbf{C}'\rangle &= -det(A)^2 tr\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\mathbf{R}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\bar{K}' & L'\\ -\bar{L}' & \bar{M}'\end{pmatrix}\right),\\
&= -det(A)^2 tr\left(\begin{pmatrix}l & 0\\ 0 & l\end{pmatrix}\begin{pmatrix}\bar{L}' & -\bar{M}'\\ \bar{K}' & L'\end{pmatrix}\right),\\
&= -det(A)^2 tr\left(\begin{pmatrix}l\bar{L}' & -l\bar{M}'\\ l\bar{K}' & lL'\end{pmatrix}\right),\\
&= -det(A)^2(l\bar{L}' + Ll) = 0.\end{aligned}$$

This implies $L' = -\bar{L}$, hence $L = r\iota, r \in \mathbb{R}$. So $\bar{A}^T\mathbf{C}'A = \mathbf{I}$. □

5. Lobachevskian Geometry

Lobachevskian geometry (Hyperbolic Geometry) is defined in the upper half plane — $\mathbb{H} = \{x + y\iota, x, y \in \mathbb{R}, y > 0\}$. It is a non-Euclidean geometry, where for every point P and line L not intersecting P, there are an infinite number of lines

through P which never cross L, [2, Section 5.1]. The Lobachevskian distance on $\mathbb{H}$ is defined as a metric which is invariant under Möbius transformations.

Definition 5.1. [2, Chapter 14] Given $z, w \in \mathbb{H}$, take C to be the circle perpendicular to the real line such that $z, w \in C$. Let $u, v \in \mathbb{R}$ be the points where C intersect the real line. The Lobachevskian distance $\rho(z, w)$ from z to w is defined as

$$\rho(z, w) = \ln([u, z, w, v]), \tag{5.1}$$

from the original Cross-ratio 2.3.

Notice that $[u, z, w, v]$ is real as u, z, v and w are concyclic. The following expressions are linked to the Lobachevskian distance [1, Thm 7.2.1].

Theorem 5.2. *For any $z, w \in \mathbb{H}$ the Lobachevskian distance satisfy:*

$$\begin{aligned}
\rho(z, w) &= \ln\left|\frac{|z-\bar{w}| + |z-w|}{|z-\bar{w}| - |z-w|}\right|,\\
\cosh^2\left(\frac{\rho(z, w)}{2}\right) &= \frac{|z-\bar{w}|^2}{4\Im(z)\Im(w)},\\
\sinh^2\left(\frac{\rho(z, w)}{2}\right) &= \frac{|z-w|^2}{4\Im(z)\Im(w)},\\
\tanh^2\left(\frac{\rho(z, w)}{2}\right) &= \frac{|z-w|^2}{|z-\bar{w}|^2}.
\end{aligned}$$

The property $\tanh^2(\rho(z, w)/2) = |z-w|^2/|z-\bar{w}|^2 = [w, \bar{z}, z, \bar{w}]$ is used to define the projective Lobachevskian distance.

Definition 5.3. Let $\underline{z}, \underline{w} \in \mathbb{P}^1(\mathbb{A})$ be essentially distinct points such that $\underline{z}$ and $\underline{\bar{w}}$ are also essentially distinct. The tanh-projective Lobachevskian distance d between $\underline{z}, \underline{w}$ is defined as

$$d(\underline{z}, \underline{w}) = \mathfrak{P}([\underline{w}, \underline{\bar{z}}, \underline{z}, \underline{\bar{w}}]). \tag{5.2}$$

The projective Lobachevskian distance $\delta(\underline{z}, \underline{w})$ is then defined by

$$\tanh^2(\delta(\underline{z}, \underline{w})/2) = d(\underline{z}, \underline{w}) \tag{5.3}$$

Note that $d(\underline{z}, \underline{w}) = r \in \mathbb{R}$ as $[\underline{\bar{w}}, \underline{z}, \underline{\bar{z}}, \underline{w}] = F_{(1\ 4)(2\ 3)}[\underline{w}, \underline{\bar{z}}, \underline{z}, \underline{\bar{w}}] = [\underline{w}, \underline{\bar{z}}, \underline{z}, \underline{\bar{w}}]$.

Remark 5.4. Given $\underline{z}, \underline{w} \in \mathbb{P}^1(\mathbb{A})$, there are the following relations for the tanh-Lobachevskian metric:

(1) It is invariant under conjugation, $d(\underline{z}, \underline{w}) = d(\underline{\bar{z}}, \underline{\bar{w}})$.
(2) It's symmetric, $d(\underline{z}, \underline{w}) = d(\underline{w}, \underline{z})$.
(3) $d(\underline{\bar{z}}, \underline{w}) = 1/d(\underline{z}, \underline{w})$.

There is an equivalent to Theorem 5.2 with the projective Lobachevskian distance.

Theorem 5.5. *For* $\underline{z}, \underline{w} \in \mathbb{P}^1(\mathbb{A})$

$$\delta(\underline{z}, \underline{w}) = \ln\left(\frac{d(\underline{z}, \underline{w})^{\frac{1}{2}} + 1}{1 - d(\underline{z}, \underline{w})^{\frac{1}{2}}}\right),$$

$$\cosh^2\left(\frac{\delta(\underline{z}, \underline{w})}{2}\right) = \frac{1}{1 - d(\underline{z}, \underline{w})},$$

$$\sinh^2\left(\frac{\delta(\underline{z}, \underline{w})}{2}\right) = \frac{d(\underline{z}, \underline{w})^{\frac{1}{2}}}{1 - d(\underline{z}, \underline{w})}.$$

Proof. As $\tanh(x/2) = (e^x - 1)/(e^x + 1)$, the values can be quickly calculated.

Then from Remark 2.5 we know $d(\underline{z}, \underline{w}) = |z - w|^2/|z - \bar{w}|$ for $\mathfrak{S}(z) = \underline{z}, \mathfrak{S}(w) = \underline{w}$. Hence we can show the similarity with Theorem 5.2 by:

$$\delta(\underline{z}, \underline{w}) = \ln\left(\frac{d(\underline{z}, \underline{w})^{\frac{1}{2}} + 1}{1 - d(\underline{z}, \underline{w})^{\frac{1}{2}}}\right) = \ln\left(\frac{\frac{|z-w|}{|z-\bar{w}|} + 1}{1 - \frac{|z-w|}{|z-\bar{w}|}}\right) = \rho(\underline{z}, \underline{w}). \tag{5.4}$$

Similar calculations give the values for $\sinh(\delta/2)$ and $\cosh(\delta/2)$. □

The Lobachevskian distance is invariant under $PGL_2(\mathbb{R})$ as shown in the next theorem.

Theorem 5.6. *For all* $A \in PGL_2(\mathbb{R})$*, and* $\underline{z}, \underline{w} \in \mathbb{P}^1(\mathbb{A})$ *there exists the following identity:* $\delta(A\underline{z}, A\underline{w}) = \delta(\underline{z}, \underline{w})$*.*

Proof. We calculate:

$$\begin{aligned}
\delta(A\underline{z}, A\underline{w}) &= \ln\left(\frac{d(A\underline{z}, A\underline{w})^{\frac{1}{2}} + 1}{1 - d(A\underline{z}, A\underline{w})^{\frac{1}{2}}}\right) \\
&= \ln\left(\frac{\mathfrak{P}([A\underline{w}, \bar{A}\bar{\underline{z}}, A\underline{z}, \bar{A}\bar{\underline{w}}])^{\frac{1}{2}} + 1}{1 - \mathfrak{P}([A\underline{w}, \bar{A}\bar{\underline{z}}, A\underline{z}, \bar{A}\bar{\underline{w}}])^{\frac{1}{2}}}\right) \\
&= \ln\left(\frac{\mathfrak{P}([A\underline{w}, A\bar{\underline{z}}, A\underline{z}, A\bar{\underline{w}}])^{\frac{1}{2}} + 1}{1 - \mathfrak{P}([A\underline{w}, A\bar{\underline{z}}, A\underline{z}, A\bar{\underline{w}}])^{\frac{1}{2}}}\right) \\
&= \ln\left(\frac{d(\underline{z}, \underline{w})^{\frac{1}{2}} + 1}{1 - d(\underline{z}, \underline{w})^{\frac{1}{2}}}\right) = \delta(\underline{z}, \underline{w}).
\end{aligned}$$

Here we used that A is a real matrix. □

Proposition 5.7. *For* $\underline{z}, \underline{w} \in \mathbb{P}^1(\mathbb{A})$ *such that* $\mathfrak{P}(\underline{z}), \mathfrak{P}(\underline{w}) \in \mathbb{H}$*,* $\delta(\underline{z}, \underline{w}) = \rho(\mathfrak{P}(\underline{z}), \mathfrak{P}(\underline{w}))$*.*

Proof. Take $\underline{z} = (x, y), \underline{w} = (u, v)$ then let $z = \mathfrak{P}(\underline{z}) = x/y, w = \mathfrak{P}(\underline{w}) = u/v$. Then

$$\begin{aligned}
\tanh^2(\rho(z, w)/2) &= \frac{|z - w|^2}{|z - \bar{w}|^2} = \frac{|x/y - u/v|^2}{|x/y - \bar{u}/\bar{v}|^2} = \frac{|xv - uy|^2/|vy|^2}{|x\bar{v} - \bar{u}y|^2/|\bar{v}y|^2} \\
&= \mathfrak{P}\begin{pmatrix} |xv - uy|^2 \\ |\bar{x}v - u\bar{y}|^2 \end{pmatrix} = \mathfrak{P}([\underline{w}, \bar{\underline{z}}, \underline{z}, \bar{\underline{w}}]) \\
&= d(\underline{z}, \underline{w}) = \tanh^2(\delta(\underline{z}, \underline{w})/2).
\end{aligned}$$

□

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References

[1] Alan F. Beardon, *The geometry of discrete groups.* 1983 (English).
[2] ———, *Algebra and geometry.* Cambridge: Cambridge University Press. xii, 326 p. 48.00, \$ 90.00/hbk; 21.99, \$ 39.99/pbk, 2005 (English).
[3] Ewain Gwynne and Matvei Libine, *On a quaternionic analogue of the cross-ratio*, Advances in Applied Clifford Algebras (2011), 1–13. 10.1007/s00006-012-0325-9.
[4] Vladimir V. Kisil, *Erlangen program at large–0: Starting with the group* $\mathrm{SL}_2(\mathbf{R})$, Notices Amer. Math. Soc. **54** (2007), no. 11, 1458–1465. E-print: `arXiv:math/0607387`, On-line. MR2361159
[5] ———, *Erlangen program at large–1: Geometry of invariants*, SIGMA, Symmetry Integrability Geom. Methods Appl. **6** (2010), no. 076, 45 pages. E-print: `arXiv:math.CV/0512416`.
[6] ———, *Geometry of Möbius transformations: Elliptic, parabolic and hyperbolic actions of* $\mathrm{SL}_2(\mathbf{R})$, Imperial College Press, 2012.
[7] I.M. Yaglom, *A simple non-Euclidean geometry and its physical basis. An elementary account of Galilean geometry and the Galilean principle of relativity. Translated from the Russian by Abe Shenitzer. With the editorial assistance of Basil Gordon.* 1979 (English).

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